3. Let us assume $a > 0$ and $c < 0$ in the Cauchy data (1.2). Then the values of c / a, defined by the inequalities

$$\Phi_2(A) \leqslant c / a \leqslant \Phi_3(A)$$

where

$$\Phi_2(A) = -[(9A^2 - 8 \operatorname{sign} s)^{1/2} + 5A] / 4A$$

$$\Phi_3(A) = \begin{cases} B / A, & A > 17\sqrt{2}/12 \\ -[(9C^2 - 8 \operatorname{sign} s)^{1/2} + 5C] / 4A, & A < 17\sqrt{2}/12 \end{cases}$$

correspond to the continuous flows.

For $A \to \infty$ $(s \to 0)$ we have $\Phi_2 \to -2$, $\Phi_3 \to -1/2$. If

$$\Phi_4(A) < c / a < 0, \qquad \Phi_4(A) = B / A.$$

then a shock wave arises and the flow velocity behind the shock decreases in the direction towards the exhaust end of the nozzle. The dependence of Φ_2, Φ_3, Φ_4 on A is shown in Fig. 1. Here the continuous flow corresponds to the region $Q_\pm$.

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REFERENCES

1. Frankl', F. I., On the theory of Laval nozzles. Izv. Akad. Nauk SSSR, Ser. Matem., Vol. 9, №5, 1945.
2. Ryzhov, O. S., Shock waves formation in Laval nozzles. PMM Vol. 27, №2, 1963.
3. Shifrin, E. G., Two-dimensional rotational flow at the neighborhood of the point of orthogonality of the sonic line to the velocity vector. Izv. Akad. Nauk SSSR, Mekhan. Zhidk. i Gaza, №6, 1966.
4. Fal'kovich, S. V., On the theory of the Laval nozzle. PMM Vol. 10, №4, 1946.
5. Il'inskaia, G. B. and Lifshits, Iu. B., On transonic flow disturbances due to vorticity. Izv. Akad. Nauk SSSR, Mekhan. Zhidk. i Gaza, №1, 1971.

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ON STABILITY OF THREE-DIMENSIONAL PERIODIC MOTIONS IN HYDRODYNAMICS

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We obtain exact conditions for the stability of periodic motions. We show that the conditions found in [1] are necessary and sufficient, but they are only applicable to motions not dependent on time. The conditions given in [2] are applicable in the general case but are only sufficient (necessary) conditions of instability (stability). We consider the dependence of stationary motions on parameters.

Systems in which the length l is much larger than the transverse dimensions, are often studied in hydrodynamics (for example, in the problem concerning the motion of the liquid between rotating cylinders the length of the cylinders is considered large in comparison with the gap between the cylinders; in the study of a positive column of a gaseous discharge the length of the column is considered large in comparison with the radius of the discharge pipe). In these cases the system end effects are neglected and it is assumed that $l = \infty$; the state of these systems is defined by the parameters of the infinite problem.

One of the possible states is the equilibrium state; it does not depend on the time t and on the longitudinal coordinate x $(-l < 2x < l)$. Infinitely small perturbations of equilibrium depend on t and x in the form $Q(t, k) \exp(ikx)$; the amplitudes Q of the perturbations are proportional to $\exp pt$, where $p(k)$ are the eigenvalues of the stability problem. For a loss of the equilibrium stability, $\gamma(k) = \operatorname{Re} p > 0$ in some interval of the wave numbers k; the amplitudes of corresponding perturbations grow exponentially.

In the case of a large, but finite length l, it is assumed [1, 2] that k takes on a discrete series of values differing by $\delta \sim 1/l$; as a result, the number of increasing perturbations is found to be finite, but arbitrarily large.

The nonlinear effects of the interaction of a large number of increasing perturbations often leads to the "survival" of only one perturbation and a suppression of perturbations with other wave numbers; as a result, there arises a stationary three-dimensional periodic motion (for example, vortices between the rotating cylinders or ionization waves in the positive discharge column). For a theoretical examination of such a motion its wave number k remains undetermined [1–4]; it may have an arbitrary value for which $\gamma(k) > 0$.

A possible way to remove this indeterminacy is through a study of the stability of the stationary motion. The stability conditions found in [1, 2] do not agree with one another. We show below that these conditions are approximate; we find exact stability conditions.

The amplitude $Q(k)$ of the unsteady motion satisfies an equation [2], which describes the interaction of the perturbations with various k values, namely,

$$dQ/dt = pQ + \sum_{m=1}^{\infty} \sum \Gamma_m Q_1 \ldots Q_{m+1} \bar{Q}_{m+2} \ldots \bar{Q}_{1+2m} \tag{1}$$

Here $p = \gamma + i\Omega$, $Q_i = Q(k_i)$ and the second sum is taken over numbers satisfying the condition

$$k_1 + \ldots + k_{m+1} - k_{m+2} - \ldots - k_{1+2m} = k, \quad k_i \approx k_0 \tag{2}$$

The functions $\Gamma_m(k_1, \ldots, k_{1+2m})$ are symmetric with respect to the first $(m+1)$ and the last m arguments. The increment of $\gamma(k) > 0$ in a region with half-width $\Delta = (-2\gamma_0 / \gamma_0'')^{1/2}$, where $\gamma_0 = \gamma(k_0)$ is the maximum value of γ. The parameter $\mu = \Delta / k_0 \ll 1$; the approximate Eq. (2) means that $k_i' = (k_i - k_0) / \Delta \sim 1$.

Henceforth we shall use only the dimensionless numbers k', therefore we omit the prime notation (for example, $\gamma / \gamma_0 \approx 1 - k^2$).

The stationary periodic motion (in which only $Q(k) \neq 0$) is described by the relations

$$Q = \sqrt{q} e^{i\theta}, \quad \theta = \theta_0 + \omega t, \quad \omega = \Omega + \sum_{m=1}^{\infty} \Omega_m q^m$$
$$-\sum_{m=1}^{\infty} \gamma_m q^m = \gamma \geqslant 0, \quad \gamma_m + i\Omega_m = p_m(k) = \Gamma_m(k, k, \ldots, k) \tag{3}$$

In the sequel we assume that $\gamma_1 < 0$, so that $q \approx -\gamma / \gamma_1$.

For infinitely small perturbations Q° we obtain from Eqs. (1) – (3)

$$dQ_+^\circ / dt = Q_+^\circ A_+^\circ + \bar{Q}_-^\circ B_+ e^{2i\theta}, \qquad \overset{\circ}{Q}_\pm = Q^\circ(k \pm \xi) \tag{4}$$

$$\overset{\circ}{A}_\pm = p(k \pm \xi) + \sum_{m=1}^{\infty} (m+1) q^m \Gamma_m \,(k \pm \xi, k, k, \ldots, k)$$

$$B_\pm = \sum_{m=1}^{\infty} m q^m \Gamma_m \quad (k, k, \ldots, k, \; k \mp \xi)$$

The second equation is obtained from Eqs. (4) by replacing ξ by $-\xi$. These equations are obtained in [1] in the approximation

$$\overset{\circ}{A}_\pm = p(k \pm \xi) + 2qp_1, \qquad B_\pm = qp_1 \tag{5}$$

The substitution

$$\overset{\circ}{Q}_\pm = R_\pm e^{i\theta}, \qquad A_\pm = \overset{\circ}{A}_\pm - i\omega \tag{6}$$

brings Eqs. (4) to the form

$$dR_+ / dt = R_+ A_+ + \bar{R}_- B_+, \quad d\bar{R}_- / dt = \bar{R}_- \bar{A}_- + R_+ \bar{B}_- \tag{7}$$

Here the second equation is obtained from the first one by replacing ξ by $-\xi$ and taking complex conjugates. The solutions of these equations are proportional to $\exp \sigma t$, where

$$\sigma = \alpha_+ \pm (\alpha_-^2 + \beta)^{1/2}, \; \alpha_\pm = {}^1/_2 \,(A_+ \pm \bar{A}_-), \; \beta = B_+ \bar{B}_- \tag{8}$$

The conjugate equations (7) correspond to conjugate values of σ The solution (3) is stable if $\operatorname{Re} \sigma < 0$ for any ξ and unstable if $\operatorname{Re} \sigma > 0$ for some ξ. Setting

$$U_\pm = \operatorname{Re} A_\pm, \; x = -{}^1/_2 \,(U_+ + U_-), \quad \chi = |\operatorname{Re} (\alpha_-^2 + \beta)^{1/2}| \tag{9}$$

$$\alpha_- = u + iv, \; \beta = r + is, \; y = U_+ U_-, \; \varepsilon = s\,(uv + {}^1/_4\, s)$$

and noting that

$$2\chi^2 = \varphi + (\varphi^2 + \psi^2)^{1/2}, \quad \varphi = u^2 - v^2 + r, \quad \psi = s + 2uv$$

we find that the stability condition $x > \chi$ is satisfied if

$$x > 0, \; f = x^2 (x^2 - \varphi) - {}^1/_4 \psi^2 = y\,(x^2 + v^2) - x^2 r - \varepsilon > 0 \tag{10}$$

For $\xi = 0$ we have $f = 0$; it follows directly from Eq. (8) that the corresponding number $\sigma = 0$ and that the other number is negative. With the help of Eqs. (7) it is easy to show that for $t \to \infty$ the perturbation with $\xi = 0$ does not vanish completely, but results only in a change of the phase θ_0 of the stationary solution (3). This means that the stationary solution is stable relative to perturbations with $\xi = 0$.

The case $\xi \neq 0$ is considered below in the approximation (5), when $s = \varepsilon = 0$. The accuracy of the approximation is given by the estimate $(s / \xi) / r \sim \varepsilon / f \sim \mu$ (it follows from the fact that the quantities Γ_m and β are expanded in series with respect to $\mu\xi$ with coefficients of the same order and that r is an even and s is an odd function of ξ). To within quantities $\sim \mu$, we obtain from Eqs. (5), (6), (8) and (9)

$$\gamma = 1 - k^2, \qquad x = \gamma + \xi^2, \qquad U_{\neq} = -x \mp 2k\xi \tag{11}$$

$$y = x^2 - 4k^2\xi^2, \qquad r = \gamma^2 (1 + \eta^2), \quad v = \rho\xi^2 - \gamma\eta$$

$$(\eta = \Omega_1 / \gamma_1, \quad \rho = -\Omega_0'' / \gamma_0'', \quad 0 \leqslant \gamma \leqslant 1)$$

Here y and r are written to within the factor γ_0^2 and the remaining terms to within γ_0 (in the inequalities (10) these factors are unessential).

According to (11), the condition $x > 0$ is satisfied everywhere. To satisfy the second of the conditions (10) it is necessary that $x^2 > \varphi$; this gives

$$\rho\eta < 1, \quad \gamma > 2 / (3 - \eta\rho) \tag{12}$$

The second necessary condition for satisfaction of the inequality $f > 0$, is $y > 0$ which holds for $\gamma > 1/2$ (the contrary inequality is a sufficient condition for instability). The latter relation can also be obtained from the inequality

$$\mathrm{Re}\,\sigma - \max(U_+, U_-) = \chi - |u| \geqslant 0$$

which is satisfied to within quantities $\sim\mu$ (if $u^2 r \geqslant -\varepsilon$). From this inequality it follows that for stability it is necessary that $U_+ < 0$; the latter is satisfied for $\gamma > 1/2$ [2]. In the particular case $\rho = \eta = 0$, the condition $\gamma > 2/3$ is necessary and sufficient for stability [1]. In the general case the condition of stability is

$$f / (\xi^2 a) = F(x) = x^3 + bx^2 + cx + d > 0 \qquad (x \geqslant \gamma)$$
$$a = 1 + \rho^2, \quad b = \gamma(3 + h) - 4, \quad c = 4\gamma(1 - \gamma)(2 - h)$$
$$d = -4\gamma^2(1 - \gamma)(g - h + 1)$$
$$h = 2(1 - \eta\rho) / a, \quad g = (1 + \eta^2) / a$$

For stability it is necessary that $F(\gamma) > 0$ which gives

$$h > 0, \quad \gamma > 1 / (1 + 1/4\, h / g) \tag{13}$$

The conditions (12) follow from the inequalities (13). When the inequalities (13) are satisfied, it is sufficient for stability that $dF / dx \geqslant 0$ for $x \geqslant \gamma$; The latter is satisfied if

$$\gamma \geqslant 4 / (\min(h, 1 / h) + 6) \tag{14}$$

or if

$$b^2 < 3c \tag{15}$$

If neither one of the conditions (14) and (15) is satisfied, then F has a minimum for $x_* > \gamma$; stability holds if

$$F(x_*) > 0 \qquad (3x_* = (b^2 - 3c)^{1/2} - b) \tag{16}$$

when the inequalities (13) are satisfied. When the condition (16) is satisfied, the polynomial F has only a single real root; therefore the condition for uniqueness of the root is

$$(b^3 - 9/2\, bc + 27/2\, d)^2 > (b^2 - 3c)^3 \tag{17}$$

which is equivalent to the condition (16).

Thus, the periodic solution (3) is stable if both of the conditions (13) and one of the conditions (14) – (17) are satisfied; it is unstable if one of the conditions (13) is not satisfied or if all the conditions (14) – (17) are not satisfied.

In particular, the solution with $\gamma = 1$ is stable if $h > 0$. when

$$\gamma_1\gamma'' + \Omega_1\Omega'' = \mathrm{Re}\,(\bar{p}_1 p'') > 0 \tag{18}$$

When $h < 0$, the solution with $\gamma = 1$ is unstable relative to perturbations for which $\xi^2 < -h$. The increment of $\operatorname{Re}\sigma$ assumes a maximum value γ_* for $\xi = \xi_*$, where

$$\xi_*{}^2 = (-h + {}^1/_4 h^2 a) \,/\, (E - {}^1/_2 ha), \quad \gamma_* = -{}^1/_2 \xi_*{}^2 \gamma_0 ha \,/E \qquad (19)$$
$$E = (1 - h + {}^1/_4 h^2 a)^{1/2} + 1$$

From this, when h is small, we obtain the estimate $\gamma_* \sim \gamma_0 h^2$ used below.

The results obtained in [1 – 6] and in the present paper can accept as true the following interpretation.

Suppose that an unstable equilibrium state corresponds to the parameters λ and initial amplitudes Q at $k \approx k_0$ are of the same order (here, and in what follows, k has the usual dimensionality). Then when the condition (18) is satisfied, there arises [2, 3] a periodic motion (*) with $k = k_0(\lambda)$. According to experiment [5], the characteristic rise time $\tau \sim l^2 / \gamma_0''$ is proportional to the square of the length l [2]. When the condition (18) is not satisfied a nonperiodic (turbulent) motion arises; its amplitudes Q_i satisfy the estimate [4]

$$\sum_i |Q_i|^2 \leqslant -\gamma_0 / \gamma_1 \qquad (k_i \approx k_0)$$

The rise time $\tau_* \sim 1 / \gamma_*$ for this motion depends on h.

Suppose that h is small, so that $(-h)^{1/2} \sim \delta \sim 1 / (l\Delta) \ll 1$, where δ is the dimensionless distance between neighboring wave numbers; then from (19) we obtain $\tau / \tau_* \sim \delta^2 \ll 1$. From the last inequality it follows that for small h the motion is periodic for $\tau \ll t \ll \tau_*$.

It was assumed above that the initial amplitudes of the perturbations are of the same order. Experimentally, such an initial condition realizes as follows. At the beginning a stable homogeneous equilibrium state corresponding to any of parameters λ_0, for which $\gamma_0(\lambda_0) < 0$, is established. Then, the parameters change rapidly (in comparison with the time τ) to values λ which correspond to an unstable equilibrium state; moreover, it is essential that the characteristic time to establish equilibrium (independently of its stability) is determined by the "transverse" dimensions of the system (the distance between planes, tube radius, etc.) and is therefore much less than τ.

When the parameters change rapidly, the motion being established does not depend on λ_0 and on the form of the curve L joining the points λ_0 and λ in the range of the parameters. The form of this curve is essential for a slow (in comparison with τ) variation of the parameters; in particular, if L intersects the stability boundary at the point λ_*, and when the condition (18) is satisfied, then a periodic motion arises with the number $k_* = k_0(\lambda_*)$. This motion may prove to be stable on the whole curve L including the point λ. It follows from the above, that when the parameters change slowly, the wave number of the possible periodic motions at the point λ depends on the form of the curve L (such a dependence was noticed in [5]).

A continuous transition of a periodic stationary motion into a nonperiodic motion was observed in [6]. Such a transition can be explaned by the fact that when the parameters vary the left side of the condition (18) changes sign.

*) The dependence of thi§ motion on λ is considered in [2].

REFERENCES

1. Echhaus, W., Studies in nonlinear stability theory. Springer, New York, 1965.
2. Ponomarenko, Iu. B., Occurrence of space-periodic motions in hydrodynamics. PMM Vol. 32, №1, 1968.
3. Newell, A. C., Lange, C. G. and Aucoin, P. J., Random convection. J. Fluid Mech., Vol. 40, Pt. 3, 1970.
4. Diprima, R. C., Eckhaus, W. and Segel, L. A., Nonlinear wave-number interaction in near-critical two-dimensional flows. J. Fluid Mech., Vol. 49, Pt. 4, 1971.
5. Snyder, H. A., Wave-number selection at finite amplitude in rotating Couette flow. J. Fluid Mech., Vol. 35, Pt. 2, 1969.
6. Zaitsev, A. A. and Shvilkin, B. N., Nature of moving strata near the boundary of their disappearance with a decrease of pressure. Radiotekhnika i Elektronika, Vol. 12, №4, 1967.

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ASYMPTOTIC ANALYSIS OF STATIONARY PROPAGATION OF THE FRONT OF A TWO-STAGE EXOTHERMIC REACTION IN A GAS

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We construct an approximate solution of the problem concerning the propagation of a planar front of a two-stage exothermic sequential chemical reaction in a gas, by the method of matched asymptotic expansions. As the parameter in the expansion we use the ratio of the adiabatic combustion temperature to the sum of the activation temperatures of both reactions. Depending on the values of the characteristic parameters of the problem, we consider several solutions, each with a different asymptotic behavior, corresponding to the various flame front propagation modes. The analytical results obtained are compared with numerical data available in the literature.

1. Formulation of the problem. The stationary propagation of a planar front of a two-stage sequential exothermic reaction $A_1 \to A_2 \to A_3$ in a gas can, subject to a number of simplifying assumptions, be described by the following equations and boundary conditions:

$$\frac{\lambda}{c}\frac{d^2T}{dx^2} - m\frac{dT}{dx} + \frac{Q_1}{c}a_1\rho k_1 \exp\frac{-E_1}{RT} + \frac{Q_2}{c}a_2\rho k_2 \exp\frac{-E_2}{RT} = 0 \quad (1.1)$$

$$\rho D\frac{d^2a_1}{dx^2} - m\frac{da_1}{dx} - a_1\rho k_1 \exp\frac{-E_1}{RT} = 0 \quad (1.2)$$

$$\rho D\frac{d^2a_2}{dx^2} - m\frac{da_2}{dx} + a_1\rho k_1 \exp\frac{-E_1}{RT} - a_2\rho k_2 \exp\frac{-E_2}{RT} = 0 \quad (1.3)$$